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Marangoni convection on an inhomogeneous substrate

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We consider an inhomogeneous substrate for Marangoni convection. The inhomogeneity shows up in a nonuniform temperature distribution which we model by a periodic variation. The response can exhibit parametric resonance. Both resonant and nonresonant responses are considered and a possible strong effect on wave number selection indicated. [S1063-651X(99)02505-2]

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I. INTRODUCTION

Marangoni convection (i.e., surface tension driven as opposed to Benard or buoyancy driven) occurs when the fluid layer is very thin and can set in with finite wave number or with zero wave number [1–4]. This is an additional feature when compared to the Benard convection and recent experiments have established the occurrence of both varieties quite dearly. From a theoretical standpoint, there is additional interest as the control parameter (Marangoni number M) for Marangoni convection occurs only in the boundary condition unlike Benard convection, where the control parameter (Rayleigh number R) occurs in the governing differential equations. The geometry envisaged is that of a thin fluid layer on an infinite conducting plate which is heated from below. The free surface at the top is ideally taken to be insulating. For a sufficiently low-temperature difference across the layer, there is no movement of the fluid, i.e., $\vec{v}=\vec{0}$ with a temperature profile $T(z)=T_1-\beta z$, where T_1 is the temperature of the conducting plate placed at $z=0$, and β is the temperature gradient. Fluctuations about this steady state are characterized by $\delta\vec{v}(\vec{r},t)$ and $\delta T(\vec{r},t)$. The fluid is taken to be incompressible. With the fluctuations made appropriately dimensionless, the z component of the velocity fluctuation is denoted by δw and the temperature fluctuation by $\delta\theta$. If the plate is infinite in extent, the translational invariance in the x-y plane will lead to oscillatory behavior in the plane and we can write $\delta w=e^{i\vec{k}\cdot\vec{r}}w(z,t)$ and $\delta\theta=e^{i\vec{k}\cdot\vec{r}}\theta(z,t)$, where $\vec{k}$ is a wave number in the two-dimensional plane. The functions w and θ satisfy the linear stability equations (for a stationary instability, i.e., $\partial/\partial t=0$)

$$(D^2-a^2)^2 w=0, \tag{1}$$

$$(D^2-a^2)\theta=-w, \tag{2}$$

where the dimensionless wave number $a=kd$, d being the mean thickness of the fluid layer and $D=d/dz$. The boundary conditions are

$$w=\frac{\partial w}{\partial z}=0 \quad \text{at} \quad z=0, \tag{3}$$

$$w=0 \quad \text{at} \quad z=1, \tag{4}$$

$$\theta=0 \quad \text{at} \quad z=0, \tag{5}$$

$$\frac{\partial\theta}{\partial z}=0 \quad \text{at} \quad z=1, \tag{6}$$

$$\left(B-\frac{\partial^2}{\partial x^2}\right)\left(D^2-\frac{\partial^2}{\partial x^2}\right)w-M\,\mathrm{Cr}\frac{\partial^2}{\partial x^2}\left(D^2+3\frac{\partial^2}{\partial x^2}\right)Dw$$

$$-M\frac{\partial^2}{\partial x^2}\left(B-\frac{\partial^2}{\partial x^2}\right)\theta=0 \quad \text{at} \quad z=1, \tag{7}$$

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the last coming from the normal and tangential force balance across the free surface [5]. In the above $M=\alpha\beta d^3 S_0/\lambda\rho\nu$ where $\alpha=(1/S_0)(\partial S/\partial T)$, $\mathrm{Cr}=\rho\nu\lambda/S_0 d$ is the Crispation number [6], and B is the Bond number given by $B=\rho d^2 g/S_0$. The mean surface tension is given by S_0, ρ is the density, ν is the kinematic viscosity, and λ is the thermal diffusivity. A solution of Eqs. (1) and (2), consistent with the boundary conditions, is obtained for

$$M=\frac{8a^2C(CS-a)(B+a^2)}{(S^3-a^3C)(B+a^2)+8a^5C\mathrm{Cr}}, \tag{8}$$

where $C=\cosh a$ and $S=\sinh a$. For the finite wave number convection Cr (which is usually a small number and less than of the order of 10^{-1}) can be dropped and critical value M_c is found to be 81.8 with $a_c\simeq 2.11$. Near $a=0$, we can expand M as

$$M=\frac{2}{3}\frac{B}{\mathrm{Cr}}\left[1+a^2\left(\frac{1}{5}+\frac{1}{B}-\frac{B}{120\mathrm{Cr}}\right)+\cdots\right]. \tag{9}$$

For $0.2+B^{-1}>B/120\mathrm{Cr}$, there is a minimum at $a=0$, and the long wavelength convection will set in if $2B/3\mathrm{Cr}<81.8$. Thus, it is possible to have both finite wave number and zero wave number onset for Marangoni convection.

In the present work, we consider a situation where the temperature of the plate at $z=0$ is not uniform. In natural situations where Marangoni convection occurs, it is likely that the temperature of the surface will be inhomogeneous and hence it could be interesting to study the effect of an inhomogeneous temperature distribution on the onset. We will consider a periodic modulation [7,8] as our inhomogeneity and study its effect on the critical Marangoni number. In the case where the instability occurs as a long wavelength roll, this effect has been studied by Tan *et al.* [9]. In their case the critical Marangoni number is less than unity and so for any resonable Marangoni number, a nonlinear analysis had to be carried out. In particular, the situation under which the surface ripples was determined. We consider a situation, where $a_c\neq 0$ and M_c is a number significantly larger than unity. This allows us to for shifts in M_c as the first effect. In particular we will show that for sufficient strength of the modulation, the inhomogeneity can cause a change from long wavelength to finite wavelength pattern in the situation where a parametric resonance occurs.

If the temperature of the bottom plate is modulated as $T_1+\Delta T\sin bx$, then the conduction state temperature profile is

$$T(x,z)=T_1-\beta z+\Delta T\frac{\cosh b(1-z)}{\cosh bd}\sin bx \tag{10}$$

and the linear stability equation for the problem becomes

$$\nabla^4 w=0, \tag{11}$$

$$\nabla^2\theta=-w+\epsilon w\cos bx. \tag{12}$$

In writing the above equations, we have made a simplification in that we have taken the amplitude ϵ of the modulating term to be a constant. In principle, because of the complicated nature of the temperature profile, the amplitude ϵ will be a function of z. However, this complication, apart from making the algebra more tedious, does not have any qualitative effect on the final result. We have checked the numerical effect of the simplification and found it to be of the order of 10%. In Sec. II, we present the effect of the modulation on the threshold Marangoni number for the two specific responses: (i) response with wave number $b/2$ (parametric resonance) and (ii) response with wave number b (typical solution without resonance).

The case of parametric resonance is by far the most important. The branching of translational symmetry in the x direction by the heating shows up in this case as a phase dependence of the response to the temperature drive. This phase dependence is analogous to that observed in the Mathieu equation—which is true for Mathieu eqation in the temporal dependence is true for our system in its x dependence. We conclude with a discussion of our results in Sec. III.

II. RESPONSE UNDER MODULATION

A. Response at wave number *b*/2

We first take up the case of parametric resonance. This is where the response occurs with a wave number $b/2$. To find the critical Marangoni number for the response, we write the velocity field (for threshold calculation, we can take the pattern to be two dimensional):

$$w=A_1g_1(z)\cos\frac{(bx+\phi)}{2}+A_3g_3(z)\cos\frac{3(bx+\phi)}{2}+\cdots, \tag{13}$$

where

$$g_1(z)=\sinh\frac{bz}{2}-\frac{bz}{2}\cosh\frac{bz}{2}+\frac{(b/2)C_{1/2}-S_{1/2}}{S_{1/2}}z\sinh\frac{bz}{2} \tag{14}$$

and

$$g_3(z)=\sinh\frac{3bz}{2}-\frac{3bz}{2}\cosh\frac{3bz}{2}+\frac{(3b/2)C_{3/2}-S_{3/2}}{S_{3/2}}z\sinh\frac{3bz}{2}. \tag{15}$$

We follow the notation that $C_m=\cosh mb$ and $S_m=\sinh mb$. The above function $g_1(z)$ and $g_3(z)$ are obtained from Eq. (1) with the boundary conditions satisfied. With w given by Eq. (13), the differential equation [Eq. (12)] for θ becomes

$$\left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial x^2}\right)\theta=-A_1\left(1-\frac{\epsilon}{2}\right)g_1(z)\cos\frac{bx}{2}\cos\frac{\phi}{2}+A_1\left(1+\frac{\epsilon}{2}\right)g_1(z)\sin\frac{bx}{2}\sin\frac{\phi}{2}$$
$$-A_3g_3(z)\cos\frac{3\phi}{2}\cos\frac{3bx}{2}+A_3g_3(z)\sin\frac{3\phi}{2}\sin\frac{3bx}{2}+\frac{\epsilon}{2}A_1g_1(z)\cos\frac{\phi}{2}\cos\frac{3bx}{2}$$
$$-\frac{\epsilon}{2}A_1g_1(z)\sin\frac{\phi}{2}\sin\frac{3bx}{2}+\frac{\epsilon}{2}A_3g_3(z)\cos\frac{\phi}{2}\cos\frac{bx}{2}-\frac{\epsilon}{2}A_3g_3(z)\sin\frac{\phi}{2}\sin\frac{bx}{2}+\cdots. \tag{16}$$

The solution for θ can be written as

$$\theta=f_1(z)\cos\frac{bx}{2}+\tilde{f}_1(z)\sin\frac{bx}{2}+f_3(z)\cos\frac{3bx}{2}+\tilde{f}_3(z)\sin\frac{3bx}{2}+\cdots, \tag{17}$$

where f_1, $\tilde{f}_1$, f_3, $\tilde{f}_3$ are the solutions of (writing $D=d/dz$)

$$\left(D^2-\frac{b^2}{4}\right)f_1=-A_1\cos\frac{\phi}{2}\left(1-\frac{\epsilon}{2}\right)g_1(z)+\frac{\epsilon}{2}A_3\cos\frac{\phi}{2}g_3(z)+\cdots, \tag{18}$$

$$\left(D^2-\frac{b^2}{4}\right)\tilde{f}_1=A_1\sin\frac{\phi}{2}\left(1+\frac{\epsilon}{2}\right)g_1(z)+\frac{\epsilon}{2}A_3\sin\frac{\phi}{2}g_3(z)+\cdots. \tag{19}$$

The solutions for f_1 and $\tilde{f}_1$ are straightforward. We require that $f_1=0$ for $z=0$ and $df_1/dz=0$ on $z=1$ and similar solutions are to be written for $\tilde{f}_1$, f_3, $\tilde{f}_3$. This yields the velocity and temperature fields in terms of the two unknown constants A_1 and A_3 and the phase ϕ. We now require that the boundary conditions of Eq. (7) be satisfied. The two harmonics need to be satisfied seperately and so also must be the sin and cos phase has to be $\phi=0$ or π. For $\phi=0$, we obtain a different set of conditions. The consistency if the two relation between A_1 and A_3 leads to

$$\left[\frac{Mb^2}{4}\frac{S_{1/2}^3-(b^3/8)C_{1/2}}{4(b^2/8)C_{1/2}S_{1/2}}\left(1\pm\frac{\epsilon}{2}\right)-\frac{2(b/2)(C_{1/2}S_{1/2}-b/2)}{S_{1/2}}\right]\left[\frac{9Mb^2}{4}\frac{S_{3/2}^3-(27b^3/8)C_{3/2}}{4(27b^3/8)C_{3/2}S_{3/2}}-2\frac{(3b/2)(C_{3/2}S_{3/2}-(3b/2))}{S_{3/2}}\right]$$
$$=\frac{\epsilon^2}{4}\frac{Mb^2}{4}\frac{9Mb^2}{4}K_{3/2}K_{1/2}, \tag{20}$$

where

$$K_{1/2}=\left[\frac{1}{2}\frac{C_{1/2}S_{1/2}-b/2}{b^3S_{1/2}}+\frac{1}{4b^3}\frac{(b/2)C_{1/2}-S_{1/2}}{S_{1/2}}C_{3/2}-\frac{S_{3/2}}{C_{3/2}}\frac{1}{b^3}\left(\frac{1}{4}\frac{(b/2)C_{1/2}-S_{1/2}}{S_{1/2}}S_{3/2}+\frac{(b^2/4)C_{1/2}^2-S_{1/2}^2}{S_{1/2}}\right)\right] \tag{21}$$

and

$$K_{3/2}=\left[\frac{3}{2}\frac{C_{3/2}S_{3/2}-3b/2}{b^3S_{3/2}}+\frac{3}{4b^3}\frac{(3b/2)C_{3/2}-S_{3/2}}{S_{3/2}}C_{1/2}-\frac{S_{1/2}}{C_{1/2}}\frac{1}{b^3}\left(\frac{3}{4}\frac{(3b/2)C_{3/2}-S_{3/2}}{S_{3/2}}S_{1/2}+\frac{(9b^2/4)C_{3/2}^2-S_{3/2}^2}{S_{3/2}}\right)\right] \tag{22}$$

and the upper sign holds for $\phi=\pi$ and the lower sign for $\phi=0$.

As expected for $\epsilon=0$, we get back the familiar result of Eq. (8). Perturbative solution of Eq. (20) in powers of ϵ, leads to

$$M=8\left(\frac{b}{2}\right)^2\frac{C_{1/2}(C_{1/2}S_{1/2}-b/2)}{S_{1/2}^3-(b/2)^3C_{1/2}}\left(1\mp\frac{\epsilon}{2}\right)+O(\epsilon^2). \tag{23}$$

The correction to M at $O(\epsilon)$ indicates the parametric resonance for a response that occurs at double the forcing length scale. Knowing that the unforced M of Eq. (8) is minimized for $a=a_c=2.1$, we can get the lowest critical Marangoni number out of Eq. (23) for $b=2a_c$. In this case,

$$M=81.4\left(1\mp\frac{\epsilon}{2}\right)+O(\epsilon^2). \tag{24}$$

The result, as we said, is dependent on the phase. The lowest possible Marangoni number will be obtained for the negative sign and hence with the phase $\phi=\pi$ for the response.

It should be noted that we have made an assumption about the two-dimensional nature of the response to the heating. In general the response will be three-dimensional which means the response would be of the form $w(x,y,z) = f(z)\cos k_1(x)\cos k_2(y)$. To get the parametric resonance we need $k_1 = b/2$. If we now go through the algebra for $k_2 \neq 0$, we would arrive at Eq. (23) with the replacements $b \to \sqrt{(b^2/4 + k_2^2)}$ and similar adjustments for $C_{1/2}$ and $S_{1/2}$. Minimization of Marangoni number now yields $\sqrt{(b^2/4 + k_2^2)} = a_c \simeq 2.1$. This means that for cylindrical rolls, $b = 2a_c$ but for other patterns with finite k_2, one would have a different relation between b and a_c. For instance, for a square roll, $k_2 = b/2$ and $b = \sqrt{2}a_c$. Thus, the rotational symmetry in the x-y plane is broken by the x-dependent heating as should be the case. The special feature of this half wave number response is best seen when we work out the critical M for other responses. It will be seen that the $O(\epsilon)$ term will be absent in those cases.

B. Response at wave number b

In this case, we write

$$w = A_1 g_1(z)\cos bx + A_2 g_2(z)\cos 2bx + \cdots, \tag{25}$$

where

$$g_1(z) = \sinh bz - bz\cosh bz + \frac{bC_1 - S_1}{S_1} z\sinh bz \tag{26}$$

and

$$g_2(z) = \sinh 2bz - 2bz\cosh 2bz + \frac{2bC_2 - S_2}{S_2} z\sinh 2bz. \tag{27}$$

We follow the notation that $C_m = \cosh mb$ and $S_m = \sinh mb$. The corresponding equation for the temperature field is

$$\left(D^2 + \frac{\partial^2}{\partial x^2}\right)\theta = -A_1 g_1(z)\cos bx - A_2 g_2(z)\cos 2bx + \epsilon\cos bx[A_1 g_1(z)\cos bx + A_2 g_2(z)\cos 2bx] + \cdots = \left(-A_1 g_1 + \frac{\epsilon}{2}A_2 g_2\right)\cos bx + \left(-A_2 g_2 + \frac{\epsilon}{2}A_1 g_1\right)\cos 2bx + \cdots. \tag{28}$$

The solution takes the form

$$\theta = Q_1 f_1(z)\cos bx + Q_2 f_2(z)\cos 2bx + \cdots, \tag{29}$$

where

$$(D^2 - b^2) f_1 = -A_1 g_1 + \frac{\epsilon}{2}A_2 g_2 \tag{30}$$

and

$$(D^2 - 4b^2) f_2 = -A_2 g_2 + \frac{\epsilon}{2}A_1 g_1. \tag{31}$$

Carrying out steps identical to those leading to Eq. (20) now yields

$$\left[M\frac{S_1^3 - b^3 C_1}{4bC_1 S_1} - \frac{2b(C_1 S_1 - b)}{S}\right] \times\left[4M\frac{S_2^3 - 8b^3 C_2}{8bC_2 S_2} - 2\times 2b\frac{C_2 S_2 - 2b}{S_2}\right] = \frac{\epsilon^2}{4}Mb^2 4Mb^2 K_1 K_2. \tag{32}$$

For $\epsilon = 0$, this yields the standard answer $M_0 = 8C_1 b^2(C_1 S_1 - b)/S_1^3 - b^3 C_1$. Perturbative solving for M for $\epsilon = 0$ yields

$$M = \frac{8C_1 b^2(C_1 S_1 - b)}{S_1^3 - b^3 C_1} + \frac{\epsilon^2 M_0^2 b^4 K_1 K_2}{4M_0[(S_2^3 - 8b^3 C_2)/8bC_2 S_2] - 4b[(C_2 S_2 - 2b)/S_2]} \times\frac{4bC_1 S_1}{S_1^3 - b^3 C_1}. \tag{33}$$

For minimization, the value of b will correspond to the value which minimizes M_0 and hence, we find

$$M = 81.4 + \epsilon^2(6.90). \tag{34}$$

Thus, we find that the conduction is stabilized against convection, a situation familiar from many other modulation problems.

III. CONCLUSION

The response at half wave number is special, because from a comparision of Eqs. (20) and (32), it should be clear that only for the half wave number case is it possible to have an $O(\epsilon)$ effect. This effect, familiar from the study of the Mathieu equation, lowers the stability boundary and we could have an onset of convection at a significantly lower value of the Marangoni number. We can think of at least one spectacular effect of this particular response. By adjusting the Bond number B and Crispation number Cr properly, we can have an onset of zero wave number convection in the modulated system. The critical Marangoni number at the onset is $2B/\mathrm{Cr}$ as mentioned in the discussion following Eq. (9). Now under modulation the condition of parametric resonance satisfied, the onset of finite wavenumber convection becomes (for ϵ positive), for nearly all B and Cr, $M_c = 81.4(1 - \epsilon/2)$ as given in Eq. (24). (It is straightforward to check that the effect of a finite Cr on the critical M_c for finite wave number response is negligible.) If the modulation amplitude ϵ exceeds a critical value ϵ_c given by

$$81.4\left(1-\frac{\epsilon_c}{2}\right)=\frac{2B}{3\mathrm{Cr}} \tag{35}$$

then the system will revert from a zero wave number to a finite wave number instability. The modulation could thus have a striking effect as mentioned in the Introduction.

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